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Unique Solvability of an Extended Hamburger Moment Problem

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R-functions are rational functions with no poles in the extended complex plane outside a given set $\{a_1,..., a_p\}$ of points on the real axis. Methods from the theory of orthogonal polynomials can be extended to R-functions. By this means the author solved an extended Hamburger moment problem: Given sequences of real numbers $\{c_n^{(i)}: n=1, 2,...\}$, $i=1,..., p$, find conditions for a distribution function ψ to exist such that

$$\int_{-\infty}^{\infty} d\psi(t)=1, \qquad \int_{-\infty}^{\infty} \frac{d\psi(t)}{(t-a_i)^m}=c_m^{(i)}, \quad m=1, 2,..., i=1,..., p.$$

In this paper these methods are extended to treat conditions for the moment problem to have a unique solution. The results are related to the classical limit point–limit circle situation. © 1987 Academic Press, Inc.

1. INTRODUCTION

In [10] we solved the following extended Hamburger moment problem (EHMP): Given p real numbers $a_1,..., a_p$ and p sequences of real numbers $\{c_m^{(i)}: m=1, 2, 3,...\}$, $i=1,..., p$, find conditions for the existence of a distribution function ψ (i.e., a bounded, real-valued, nondecreasing function with infinitely many points of increase) such that

$$\int_{-\infty}^{\infty} d\psi(t)=1, \qquad \int_{-\infty}^{\infty} \frac{d\psi(t)}{(t-a_i)^m}=c_m^{(i)}, \quad m=1, 2,..., i=1,..., p. \tag{1.1}$$

The problem was treated in the context of a theory of orthogonal R-functions, where R-functions are rational functions with no poles in the extended complex plane outside the set $\{a_1,..., a_p\}$. It was shown that a

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necessary and sufficient condition for the existence of solution is positivity of the functional Φ defined on the space $\mathscr{R}$ of R-functions by

$$\Phi\left(\alpha_0+\sum_{i=1}^{p}\sum_{m=1}^{N_i}\frac{\alpha_{im}}{(t-a_i)^m}\right)=\alpha_0+\sum_{i=1}^{p}\sum_{m=1}^{N_i}\alpha_{im}c_m^{(i)}. \tag{1.2}$$

A central role in the proof was played by a Gaussian quadrature formula associated with the zeros of the orthogonal and quasi-orthogonal R-functions $Q_n(t)$ and $Q_n(t,\tau)$, together with Helly's selection theorem.

This result and method of proof is analogous to results and methods concerning solution of the classical Hamburger moment problem (HMP), see, e.g., [1, 4, 10], and of the strong Hamburger moment problem (SHMP), see, e.g., [5, 6, 7, 8].

The uniqueness problem for the EHMP is the following: Find conditions for the sequences $\{c_m^{(i)}\}$ to determine a unique distribution function ψ satisfying (1.1). In the classical case (HMP) the situation is as follows: Let $\{Q_n(z)\}$ be the orthogonal polynomials determined by a given positive definite sequence $\{c_m\}$ (cf., e.g., [1, 2, 3]), and let $Q_n(z,\tau)$ be the quasi-orthogonal polynomial $Q_n(z)-\tau Q_{n-1}(z)$, $\tau\in\mathbb{R}$. Further let $P_n(z,\tau)$ be the polynomial associated with $Q_n(z,\tau)$. For every index n and every fixed complex number z outside the real axis, the mapping $\tau\to P_n(z,\tau)/Q_n(z,\tau)$ maps the real axis onto the boundary of a disc $\Delta_n(z)$. The sequence of discs $\{\Delta_n(z)\}$ is nested, and the intersection $\Delta(z)$ is a disc for all z (the limit circle case) or a single point for all z (the limit point case). Let $\hat{\psi}$ denote the Stieltjes transform of ψ, i.e.,

$$\hat{\psi}(z)=\int_{-\infty}^{\infty}\frac{d\psi(t)}{t-z}, \tag{1.3}$$

and set $\sum_z=\{\hat{\psi}(z)\colon \psi \text{ is a solution of HMP}\}$. Then $\Delta(z)=\sum_z$, and thus HMP has a unique solution exactly in the limit point case. (See [1, 4, 11].) In the SHMP case the situation is completely analogous, see [6, 9].

In this paper we work out an analogous theory for the EHMP. We introduce R-functions $P_n(z,\tau)$ associated with $Q_n(z,\tau)$. By the aid of formulas of Christoffel–Darboux type we show that for every regular $Q_n(z)$ (see Sect. 2) and z outside the real axis, the mapping $\tau\to P_n(z,\tau)/Q_n(z,\tau)$ maps the real axis onto the boundary of a disc $\Delta_n(z)$. The sequence $\{\Delta_n(z)\}$ is nested, and the intersection is either a disc for every z, or a single point for every z. Set $\sum_z=\{\hat{\psi}(z)\colon \psi \text{ is a solution of EHMP}\}$, $\sum_z^0=\{\hat{\psi}(z)\colon \psi \text{ is a solution of the relaxed EHMP}\}$. (A distribution function ψ solves the relaxed EHMP if $\int_{-\infty}^{\infty} d\psi(t)/(t-a_i)^m=c_m^{(i)}$, $m=1,2,\ldots$, $i=1,\ldots,p$, while the condition $\int_{-\infty}^{\infty} d\psi(t)=1$ is omitted.) Then in general $\sum_z\subset\Delta(z)\subset\sum_z^0$, and thus unique solvability of the relaxed EHMP implies the limit point situation, while the limit point situation implies unique solvability of EHMP. Furthermore, $\sum_z = \Delta(z)$ when there are infinitely many $Q_n(z)$ that are regular and not bi-degenerate, and thus in this case the EHMP has a unique solution exactly in the limit point case.

2. Preliminaries

Let $a_1,..., a_p$ be given (fixed) real numbers. Let $\mathscr{R}$ denote the linear space consisting of all functions of the form

$$R(t) = \alpha_0 + \sum_{i=1}^{p} \sum_{j=1}^{N_i} \frac{\alpha_{ij}}{(t-a_i)^j}, \qquad \alpha_0, \alpha_{ij} \in \mathbb{C}. \tag{2.1}$$

Elements of $\mathscr{R}$ are called R-functions. We shall denotey by $\mathscr{R}_{\mathbb{R}}$ the real space of all R-functions with real coefficients.

We note that a function R belongs to $\mathscr{R}$ iff it can be written in the form $R(t) = P(t)/Q(t)$, where Q is a polynomial with all its zeros among the points $a_1,..., a_p$, and where P is a polynomial with $\deg P \leqslant \deg Q$. We call R *degenerate* if $\deg P < \deg Q$. This is the case exactly when $\alpha_0 = 0$ in the expression (2.1).

We write $\mathscr{R}(s_1,..., s_p)$ for the space of all R-functions of the form

$$R(t) = \frac{P(t)}{(t-a_1)^{s_1} \cdots (t-a_p)^{s_p}}, \qquad \deg P \leqslant s_1 + \cdots + s_p. \tag{2.2}$$

Every natural number n has a unique decomposition $n = p \cdot q_n + r_n$, $1 \leqslant r_n \leqslant p$. We write $r = r_n$, $q = q_n$ when there is no danger of confusion. We write $\mathscr{R}_n$ for $\mathscr{R}(s_1,..., s_p)$ in the special case when $s_1 = \cdots = s_r = q+1$, $s_{r+1} = \cdots = s_p = q$. We denote by $\mathscr{R}^0$ (resp. $\mathscr{R}_n^0$, $\mathscr{R}^0_{(s_1,...,s_p)}$) the subspace consisting of the degenerate R-functions in $\mathscr{R}$ (resp. $\mathscr{R}_n$, $\mathscr{R}(s_1,..., s_p)$).

In the following Φ shall denote a (fixed) linear functional on $\mathscr{R}$. This functional gives rise to a bilinear form $\langle \cdot, \cdot \rangle$, on $\mathscr{R} \times \mathscr{R}$, defined by $\langle A, B \rangle = \Phi(A \cdot B)$. (Note that $\mathscr{R}$ and $\mathscr{R}_{\mathbb{R}}$ are closed under multiplication.) We shall assume that Φ, and hence $\langle \cdot, \cdot \rangle$, is positive definite, i.e., that $\Phi(R^2) > 0$ when $R \in \mathscr{R}_{\mathbb{R}}$, $R(t) \not\equiv 0$.

By applying the Gram–Schmidt orthonormalization process (with respect tot the inner product defined above on $\mathscr{R}_{\mathbb{R}}$) to the sequence

$$1, \frac{1}{(t-a_1)},..., \frac{1}{(t-a_p)}, \frac{1}{(t-a_1)^2},..., \frac{1}{(t-a_p)^2}, \frac{1}{(t-a_1)^3},... \tag{2.3}$$

(in the indicated order), we obtain an orthonormal sequence $\{Q_n : n = 0, 1,...\}$ of functions in $\mathscr{R}_{\mathbb{R}}$. We note that $\{Q_0,..., Q_{n-1}\}$ is a base for $\mathscr{R}_{n-1}$, and thus $\langle Q_n, P \rangle = 0$ for every $P \in \mathscr{R}_{n-1}$.

For every index n and every real number τ we define the *quasi-orthogonal R-function* $Q_n(t, \tau)$ in the following way:

$$Q_n(t, \tau) = Q_n(t) - \tau \frac{(t - a_{r-1})}{(t - a_r)} Q_{n-1}(t). \tag{2.4}$$

(Here and in the following a_{r+1} means a_1 if $r = p$, and a_{r-1} means a_p if $r = 1$.) In particular $Q_n(t, 0) = Q_n(t)$. The function $Q_n(t, \tau)$ belongs to $\mathcal{R}_n$, and we may write

$$Q_n(t, \tau) = \frac{B_n(t, \tau)}{(t - a_1)^{q+1} \cdots (t - a_r)^{q+1} (t - a_{r+1})^q \cdots (t - a_p)^q}, \tag{2.5}$$

where $B_n(t, \tau)$ is a polynomial of degree at most n. (We always consider $Q_n(t, \tau)$ as a function of t for fixed τ when making statements $Q_n(t, \tau) \in \mathcal{R}_n$, $\deg B_n(t, \tau) \leqslant n$ etc...) Note that $B_n(a_r, 0) \neq 0$. We denote by $\mu(\tau)$ the degree of $B_n(t, \tau)$ and by $\nu(\tau)$ the number of zeros of $Q_n(t, \tau)$.

For further reference we state a more precise result on the structure of the functions $Q_n(t, \tau)$. A proof can be found in [10, Theorem 2.9].

THEOREM 2.1. *The function $Q_n(t, \tau)$ can be written*

$$Q_n(t, \tau) = \frac{C_n(t, \tau)}{(t - a_1)^{\kappa_1(\tau)} \cdots (t - a_p)^{\kappa_p(\tau)}}, \tag{2.6}$$

where $C_n(t, \tau) = k_n (t - t_1(\tau)) \cdots (t - t_{\nu(\tau)}(\tau))$, $t_j(\tau) \neq a_i$, $i = 1, ..., p$, $j = 1, ..., \nu(\tau)$, $t_i(\tau) \neq t_j(\tau)$ *when* $i \neq j$. *Here* $q \leqslant \kappa_i(\tau) \leqslant q + 1$ *for* $i \leqslant r$, $q - 1 \leqslant \kappa_i(\tau) \leqslant q$ *for* $i > r$, $\kappa_r(0) = q + 1$. *Furthermore* $n - 1 \leqslant \mu(\tau) \leqslant n$, $\mu(\tau) - p \leqslant \nu(\tau) \leqslant \mu(\tau)$, $\mu(0) - p + 1 \leqslant \nu(0) \leqslant \mu(0)$, $\kappa_1(\tau) + \cdots + \kappa_p(\tau) = n + \nu(\tau) - \mu(\tau)$.

The R-function $Q_n(t, \tau)$ is degenerate iff $\mu(\tau) = n - 1$. We shall call $Q_n(t)$ *bi-degenerate* if both $Q_n(t)$ and $Q_{n-1}(t)$ are degenerate. We observe that $Q_n(t, \tau)$ is degenerate for exactly one value of τ if $Q_{n-1}(t)$ is non-degenerate. If $Q_{n-1}(t)$ is degenerate, then $Q_n(t, \tau)$ is degenerate iff $Q_n(t)$ is degenerate. Thus if $Q_n(t)$ is not bi-degenerate, then $Q_n(t, \tau)$ is either non-degenerate for all τ (if $Q_{n-1}(t)$ is degenerate) or degenerate for exactly one value of τ (if $Q_{n-1}(t)$ is nondegenerate). For further use we state the following result on the "density" of non-degenerate orthogonal R-functions $Q_n(t)$. A proof can be found in [10, Proposition 3.4].

THEOREM 2.2. *Every segment $\{Q_s(t), ..., Q_{s+2p}(t)\}$ contains at least two non-degenerate elements, and thus there are infinitely many non-degenerate $Q_n(t)$.*

We call $Q_n(t,\tau)$ a_i-*defective* if a_i is a zero of $B_n(t,\tau)$, i.e., if $\kappa_i(\tau)=q$ when $i\leqslant r$, if $\kappa_i(\tau)=q-1$ when $i>r$. The orthogonal R-function $Q_n(t)$ can not be a_r-defective. We shall call $Q_n(t)$ *regular* if $Q_n(t)$ is not a_{r-1}-defective.

For further use we shall prove a couple of results on regular orthogonal R-functions $Q_n(t)$, including a "density" result.

LEMMA 2.3. *Let $Q_n(t)$ be a_{r-s}-defective, $1\leqslant s\leqslant p-1$. Then we may write*

$$Q_n(t)=\frac{(t-a_{r-s})}{(t-a_r)}\sum_{j=n-s}^{n-1}\alpha_j Q_j(t),\qquad \alpha_j\in\mathbb{R}. \tag{2.7}$$

Proof. The function $((t-a_r)/(t-a_{r-1}))\,Q_n(t)$ belongs to $\mathscr{R}_{n-1}$. Therefore we may write $((t-a_r)/(t-a_{r-1}))\,Q_n(t)=\sum_{j=0}^{n-1}\alpha_j Q_j(t)$. Let $j<n-s$. Then $((t-a_r)/(t-a_{r-s}))\,Q_j(t)\in\mathscr{R}_{n-s}$, hence $\Phi((t-a_r)/(t-a_{r-s}))\,Q_n(t)\,Q_j(t))=0$, which means $\alpha_j=0$. Consequently $((t-a_r)/(t-a_{r-s}))\,Q_n(t)=\sum_{j=n-s}^{n-1}\alpha_j Q_j(t)$. ∎

THEOREM 2.4. *The R-function $Q_n(t)$ is nonregular iff it can be expressed in the form*

$$Q_n(t)=k\frac{(t-a_{r-1})}{(t-a_r)}Q_{n-1}(t). \tag{2.8}$$

Proof. Clearly the function $Q_n(t)$ is nonregular (i.e., a_{r-1}-degenerate) if it has the form (2.8). On the other hand, if $Q_n(t)$ is nonregular, then by Lemma 2.3 $Q_n(t)$ can be written in the form (2.8). ∎

THEOREM 2.5. *Every segment $\{Q_{s+1}(t),\ldots,Q_{s+p}(t)\}$ contains at least one regular element, and thus there are infinitely many regular $Q_n(t)$.*

Proof. Assume that all the elements $Q_{s+1}(t),\ldots,Q_{s+p}(t)$ are nonregular. Let $s=r+pq$. Then by Theorem 2.4 we may write

$$\begin{aligned}Q_{s+p}(t)&=k_1\frac{(t-a_{r-1})}{(t-a_r)}Q_{s+p-1}(t)\\&=k_2\frac{(t-a_{r-1})}{(t-a_r)}\cdot\frac{(t-a_{r-2})}{(t-a_{r-1})}Q_{s+p-2}\\&=\cdots=k_p\frac{(t-a_{r-1})\cdots(t-a_{r-p+1})(t-a_r)}{(t-a_r)\cdots(t-a_{r-p+2})(t-a_{r-p+1})}Q_s(t)=k_pQ_s(t),\end{aligned}$$

which is impossible. ∎

Let n and τ be given, and set $\nu = \nu_n(\tau)$. Let $t_1, \ldots, t_\nu$ denote the zeros of $Q_n(t, \tau)$. We define the fundamental interpolating R-functions $L_{n,i}$ by

$$L_{n,i}(t) = \frac{Q_n(t, \tau)(t-a_r)}{Q_n'(t_i, \tau)(t-t_i)(t_i-a_r)}, \qquad i = 1, 2, \ldots, \nu, \tag{2.9}$$

$$L_{n,0}(t) = \frac{1}{A_n} Q_n(t, \tau) \cdot (t-a_r), \qquad \text{where} \quad A_n = \lim_{t \to \pm\infty} tQ_n(t, \tau). \tag{2.10}$$

We observe that $L_{n,i} \in \mathscr{R}_{n-1}$, $i = 0, 1, \ldots, \nu$, $L_{n,0} \equiv 0$ when $Q_n(t, \tau)$ is non-degenerate, $\lim_{t \to \pm\infty} L_{n,i}(t) = 0$ for $i = 1, \ldots, \nu$, when $Q_n(t, \tau)$ is degenerate $L_{n,i}(t_j) = \delta_{ij}$ for $i = 1, \ldots, \nu$, $j = 1, \ldots, \nu$, $L_{n,0}(t_j) = 0$ for $j = 1, \ldots, \nu$, $L_{n,0}(\infty) = 1$ when $Q_n(t, \tau)$ is degenerate. We also note that $L_{n,i}$ is degenerate iff $Q_n(t, \tau)$ is degenerate, for $i = 1, \ldots, \nu$, while $L_{n,0}$ is non-degenerate when $Q_n(t, \tau)$ is degenerate. We shall write $\lambda_{n,i} = \lambda_{n,i}(\tau)$ for $\Phi(L_{n,i})$. In particular $\lambda_{n,0} = 0$ when $Q_n(t, \tau)$ is non-degenerate.

Let f be a given R-function. We write $f(\infty)$ for the limit $\lim_{t \to \pm\infty} f(t)$, which always exists as a finite value. In particular $f(\infty) = 0$ iff f is degenerate. We define the interpolating R-function $F(t) = F_n(t, \tau)$ by

$$F(t) = F_n(t, \tau) = f(\infty) \cdot L_{n,0}(t) + \sum_{i=1}^{\nu} f(t_i)\, L_{n,i}(t). \tag{2.11}$$

Then $F \in \mathscr{R}_{n-1}$, $F(t_j) = f(t_j)$, $j = 1, \ldots, \nu$, and $F(\infty) = f(\infty)$ when $Q_n(t, \tau)$ is degenerate.

We shall now prove a quadrature formula for $\Phi(f)$ with nodes at $t_1, \ldots, t_\nu$ (and ∞ when $Q_n(t, \tau)$ is degenerate). Corresponding formulas are proved in [10, Sect. 4] for the case that $Q_n(t, \tau)$ is non-degenerate, and for the case that $Q_n(t, \tau)$ is degenerate when f is also degenerate. We shall here give a complete proof of the general formula.

In the following $\kappa_1(\tau), \ldots, \kappa_p(\tau)$ shall have the same meaning as in Theorem 2.1.

THEOREM 2.6. *The quadrature formula*

$$\Phi(f) = \lambda_{n,0} f(\infty) + \sum_{i=1}^{\nu} \lambda_{n,i} f(t_i) \tag{2.12}$$

is valid for every $f \in \mathscr{R}(s_1, \ldots, s_p)$, *where* $s_i = \kappa_i(\tau) + q + 1$ *when* $i < r$, $s_r = 2q$ *when* $\tau \neq 0$, $s_r = 2q + 1$ *when* $\tau = 0$, $s_i = \kappa_i(\tau) + q$ *when* $i > r$.

Proof. Let $f \in \mathscr{R}(s_1, \ldots, s_p)$. The function $g = f - F$ (where F is defined as above) also belongs to $\mathscr{R}(s_1, \ldots, s_p)$. We note that g is degenerate if $Q_n(t, \tau)$

is degenerate, because $F(\infty) = f(\infty)$ in this case. Since $g(t_i) = 0$, $i = 1, \dots, v$, we may write

$$g(t) = \frac{(t-t_1)\cdots(t-t_v)}{(t-a_1)^{\kappa_1(\tau)}\cdots(t-a_r)^{\kappa_r(\tau)}\cdots(t-a_p)^{\kappa_p(\tau)}}\cdot h(t), \tag{2.13}$$

with

$$h(t) = \frac{G(t)}{(t-a_1)^{q+1}\cdots(t-a_r)^{\sigma}\cdots(t-a_p)^{q}}, \tag{2.14}$$

where G is a polynomial, $\sigma = q-1$ for $\tau \neq 0$, $\sigma = q$ for $\tau = 0$. Since $(t-t_1)\cdots(t-t_v)/(t-a_1)^{\kappa_1(\tau)}\cdots(t-a_p)^{\kappa_p(\tau)} = kQ_n(t,\tau)$ and $g(t)$ is degenerate if $Q_n(t,\tau)$ is degenerate, we conclude that $h(t)$ is an R-function. It follows that $h(t) \in \mathscr{R}_{n-1}$, and $((t-a_{r-1})/(t-a_r))\,h(t) \in \mathscr{R}_{n-2}$ when $\tau \neq 0$. Consequently $\Phi(f) - \Phi(F) = \Phi(Q_n(t,\tau)\cdot h(t)) = \Phi(Q_n(t)\cdot h(t)) - \tau\Phi((t-a_{r-1})/(t-a_r))\,h(t)\,Q_{n-1}(t)) = 0$, and so $\Phi(f) = \Phi(F) = f(\infty)\,\lambda_{n,0} + \sum_{i=1}^{v} f(t_i)\,\lambda_{n,i}$. ∎

THEOREM 2.7. *The constants $\lambda_{n,i} = \lambda_{n,i}(\tau)$, $i = 1, \dots, v$, are positive, and $\lambda_{n,0}$ is positive when $Q_n(t,\tau)$ is degenerate.*

Proof. We may write $L_{n,i}(t) = D_{n,i}(t)/(t-a_1)^{\kappa_1(\tau)}\cdots(t-a_r)^{\kappa_r(\tau)-1}\cdots(t-a_p)^{\kappa_p(\tau)}$, where $\deg D_{n,i} = v-1$ for $i = 1, \dots, v$, and for $i = 0$ when $Q_n(t,\tau)$ is degenerate. It follows that the functions $\phi_{n,i}(t) = L_{n,i}(t)^2 - L_{n,i}(t)$ belong to $\mathscr{R}(2\kappa_1(\tau), \dots, 2q, \dots, 2\kappa_p(\tau)) \subset \mathscr{R}(s_1, \dots, s_p)$ for $i = 0, \dots, v$. Note that $\phi_{n,i}(t_j) = 0$, $i = 1, \dots, v$, $j = 1, \dots, v$, $\phi_{n,i}(\infty) = 0$ when $Q_n(t,\tau)$ is degenerate, for $i = 1, \dots, v$, $\phi_{n,0}(t_j) = 0$, $j = 1, \dots, v$, $\phi_{n,0}(\infty) = 0$ when $Q_n(t,\tau)$ is degenerate. By applying Theorem 2.6 to $\phi_{n,i}$ we get $\Phi(\phi_{n,i}) = \lambda_{n,0}\phi_{n,i}(\infty) + \sum_{j=1}^{v}\lambda_{n,j}\phi_{n,i}(t_j) = 0$, $i = 0, 1, \dots, v$. Thus $\lambda_{n,i} = \lambda_{n,i}(\tau) = \Phi(L_{n,i}) = \Phi(L_{n,i}^2) > 0$. ∎

3. ASSOCIATED R-FUNCTIONS AND FORMULAS OF CHRISTOFFEL–DARBOUX TYPE

Let $R(t,z)$ be an R-function in t, i.e., $R(t,z) = \alpha_0 + \sum_{i=1}^{p}\sum_{j=1}^{N_i}(\alpha_{ij}(z)/(t-a_i)^j)$. To emphasize that the functional Φ is operating on t in $R(t,z)$ we denote it by Φ_t. Thus if Φ is determined by the sequences $\{c_j^{(i)}\}$, $i = 1, \dots, p$, then

$$\Phi_t\left(\alpha_0 + \sum_{i=1}^{p}\sum_{j=1}^{N_i}\frac{\alpha_{ij}(z)}{(t-a_i)^j}\right) = \alpha_0 + \sum_{i=1}^{p}\sum_{j=1}^{N_i}\alpha_{ij}(z)\,c_j^{(i)}. \tag{3.1}$$

For every $z \in \mathbb{C}$ let D_z denote the difference quotient operator defined on $\mathscr{R}$ by

$$(D_z R)(t) = \frac{D(t) - R(z)}{t - z}. \tag{3.2}$$

We note that for every $j \geqslant 1$ we can write

$$\left[\frac{1}{(t-a_i)^j} - \frac{1}{(z-a_i)^j}\right] \cdot \frac{1}{(t-z)} = \frac{(z-a_i)^j - (t-a_i)^j}{(t-z)(t-a_i)^j (z-a_i)^j}$$
$$= \frac{S_{j-1}(t, z)}{(t-a_i)^j (z-a_i)^j},$$

where $S_{j-1}(t, z)$ is a polynomial of degree $j-1$ both in t and in z. It follows that if $R(t) \in \mathscr{R}(s_1, ..., s_p)$, then $(D_z R)(t)$ belongs to $\mathscr{R}^0(s_1, ..., s_p)$ both as a function of t and as a function of z. Consequently $\Phi_t((D_z(t))$ is defined for every z, and we set $(AR)(z) = \Phi_t((D_z R)(t))$. We observe that $AR \in \mathscr{R}^0(s_1, ..., s_p)$. We may call AR the *R-function associated with R.*

For every quasi-orthogonal R-function $Q_n(z, \tau)$ and more generally for every R-function $Q_n(z, \tau) = Q_n(z) - \tau((z - a_{r-1})/(z - a_r))\, Q_{n-1}(z)$, $\tau \in \mathbb{C}$, the associated R-function $P_n(z, \tau)$ is then given for $n = 1, 2, ...,$ by

$$P_n(z, \tau) = \Phi_t\left(\frac{Q_n(t, \tau) - Q_n(z, \tau)}{t - z}\right). \tag{3.3}$$

We set $P_0(z, \tau) = 0$. It follows from the foregoing that $P_n(z, \tau) \in \mathscr{R}_n^0$. We write $P_n(z)$ for $P_n(z, 0)$.

LEMMA 3.1. *Let P be of the form $P(z) = (z - \alpha)\, S_{n-1}(z)$, where $S_{n-1} \in \mathscr{R}_{n-1}$. Then*

$$\Phi_t([D_z(PQ_n)](t)) = P(z)\, P_n(z). \tag{3.4}$$

Proof. We may write

$$\begin{aligned}
&\Phi_t([D_z(PQ_n)](t)) \\
&\quad = P(z)\, \Phi_t((D_z Q_n)(t)) + \Phi_t\left(\frac{P(t) - P(z)}{t - z} \cdot Q_n(t)\right) \\
&\quad = P(z)\, P_n(z) + \Phi_t\left(\frac{P(t) - P(z)}{t - z} \cdot Q_n(t)\right).
\end{aligned}$$

The function P can be written $P(z) = T(z)/N(z)$, where $\deg T \leqslant n$, $\deg N \leqslant n - 1$. Consequently the function $\psi(t) = (P(t) - P(z)/(t - z))$ can be written as $\psi(t)=\phi(t)/N(t)\,N(z)$, where $\phi(t)=(T(t)\,N(z)-N(t)\,T(z))/(t-z)$ is a polynomial in t of degree at most $n-1$. It follows that $\psi(t)\in\mathscr{R}_{n-1}$, and consequently $\Phi_t(\psi(t)\,Q_n(t))=0$. ∎

LEMMA 3.2. *The following formula holds for* $n=p, p+1,...,$

$$P_n(z,\tau)=P_n(z)-\tau\frac{(z-a_{r-1})}{(z-a_r)}P_{n-1}(z). \tag{3.5}$$

Proof. From the equality $Q_n(z,\tau)=Q_n(z)-\tau((z-a_{r-1})/(z-a_r))\,Q_{n-1}(z)$ follows

$$P_n(z,\tau)=\Phi_t((D_zQ_n)(t))$$
$$-\tau\Phi_t\left(\frac{1}{(t-z)}\left[\frac{(t-a_{r-1})}{(t-a_r)}Q_{n-1}(t)-\frac{(z-a_{r-1})}{(z-a_r)}Q_{n-1}(z)\right]\right).$$

The first term is $P_n(z)$. It follows from Lemma 3.1 that the second term equals $-\tau((z-a_{r-1})^n/(z-a_r))\,P_{n-1}(z)$ when $1/(z-a_r)\in\mathscr{R}_{n-2}$, which is the case when $n\geqslant p$. ∎

We now prove a result that shows how the values of $Q_n(\zeta)$ can be determined by the values of $Q_n(z)$, $P_n(z)$ and the foregoing values of $Q_i(\zeta)$, $P_i(z)$, $Q_i(z)$, cf. [1, p. 18]. It can be used in the proof of the invariability of the limit point versus limit circle situation, see Theorem 4.2.

LEMMA 3.3. *The following formula holds for* $n=2, 3,...,$

$$(\zeta-a_r)\,Q_n(\zeta)$$
$$=(z-a_r)\,Q_n(z)+(z-a_r)(\zeta-z)\sum_{i=1}^{n-1}[P_n(z)\,Q_i(z)-Q_n(z)\,P_i(z)]\,Q_i(\zeta). \tag{3.6}$$

Proof. The function $f(\zeta)=((\zeta-a_r)/(\zeta-z))[Q_n(\zeta)-Q_n(z)]$ belongs to $\mathscr{R}_{n-1}$. We can therefore write $f(\zeta)=\sum_{i=0}^{n-1}a_i(z)\,Q_i(z)$, where

$$a_i(z)=\langle f, Q_i\rangle=\Phi_t\left(\frac{Q_n(t)-Q_n(z)}{t-z}(t-a_r)\,Q_i(t)\right)$$
$$=(z-a_r)\,Q_i(z)\,\Phi_t\left(\frac{Q_n(t)-Q_n(z)}{t-z}\right)$$
$$+\Phi_t\left(\frac{(t-a_r)\,Q_i(t)-(z-a_r)\,Q_i(z)}{t-z}[Q_n(t)-Q_n(z)]\right).$$

Since the function $g(t)=((t-a_r)\,Q_i(t)-(z-a_r)\,Q_i(z))/(t-z)$ belongs to

$\mathscr{R}_{n-1}$, we get $\Phi_t(g(t)\,Q_n(t))=0$, and furthermore $\Phi_t(g(t)\cdot Q_n(z))=(z-a_r)\,Q_n(z)\,P_i(z)$ for $i>0$ by Lemma 3.1. Thus $a_i(z)=(z-a_r)[Q_i(z)\,P_n(z)-Q_n(z)\,P_i(z)]$, $a_0(z)=(z-a_r)\,P_n(z)-Q_n(z)$. Hence $(\zeta-a_r)/(\zeta-z)[Q_n(\zeta)-Q_n(z)]=(z-a_r)\sum_{i=1}^{n-1}[P_n(z)\,Q_i(z)-Q_n(z)\,P_i(z)]\,Q_i(\zeta)$. ∎

We shall now prove a formula which is analogous to the Liouville–Ostrogradski formula in the theory of differential equations, and to similar formulas for othogonal polynomials and orthogonal Laurent polynomials. Next we prove formulas analogous to the Christoffel–Darboux formulas and related formulas for orthogonal polynomials and similar formulas for orthogonal Laurent polynomials (cf. [1, 2, 3, 8, 9]).

We shall write $Q_n(z)$ in the form

$$Q_n(z)=\beta_0^{(n)}+\frac{\beta_1^{(n)}}{(z-a_1)}+\cdots+\frac{\beta_{n-1}^{(n)}}{(z-a_{r-1})^{q+1}}+\frac{\beta_n^{(n)}}{(z-a_r)^{q+1}}. \tag{3.7}$$

We shall always assume that z and ζ are complex numbers different from $a_1,\ldots, a_p$.

THEOREM 3.4. *The following formula is valid for* $n=2, 3,\ldots,$

$$Q_{n-1}(z)\,P_n(z)-Q_n(z)\,P_{n-1}(z)=\frac{\beta_{n-1}^{(n)}\,(a_r-a_{r-1})}{\beta_{n-1}^{(n-1)}\,(z-a_r)(z-a_{r-1})}. \tag{3.8}$$

Proof. We may write

$$\begin{aligned} &Q_{n-1}(z)[Q_n(t)-Q_n(z)]-Q_n(z)[Q_{n-1}(t)-Q_{n-1}(z)]\\ &\quad=Q_{n-1}(t)[Q_n(t)-Q_n(t)]-Q_n(t)[Q_{n-1}(t)-Q_{n-1}(z)]. \end{aligned} \tag{3.9}$$

Application of Φ_t to the left side of (3.9) gives the left side of (3.8). The function $\phi_{n-1}(t)=(D_zQ_{n-1})(t)$ belongs to $\mathscr{R}_{n-1}$, and hence $\Phi_t(Q_n(t)(D_zQ_{n-1})(t))=0$. The function $\phi_n(t)=(D_zQ_n)(t)$ can be written

$$\begin{aligned} \phi_n(t)&=\frac{\beta_n^{(n)}}{(t-z)}\left[\frac{1}{(t-a_r)^{q+1}}-\frac{1}{(z-a_r)^{q+1}}\right]\\ &\quad+\frac{\beta_{n-1}^{(n)}}{(t-z)}\left[\frac{1}{(t-a_{r-1})^{q+1}}-\frac{1}{(z-a_{r-1})^{q+1}}\right]+\cdots\\ &=-\frac{\beta_n^{(n)}}{(t-a_r)^{q+1}(z-a_r)}-\frac{\beta_{n-1}^{(n)}}{(t-a_{r-1})^{q+1}(z-a_{r-1})}+\cdots, \end{aligned}$$

where the left-out terms as functions of t belong to $\mathscr{R}_{n-2}$. Comparison of coefficients for $1/(t-a_{r-1})^{q+1}$ in the expansion $\beta_{n-1}^{(n)}/(t-a_{r-1})^{q+1}(z-a_{r-1})=\sum_{i=0}^{n-1}\alpha_i(z)\,Q_i(t)$ and for $1/(t-a_{r-1})^{q+1}$ and for $1/(t-a_r)^{q+1}$ in the expansion $\beta_n^{(n)}/(t-a_r)^{q+1}(z-a_r)=\sum_{i=0}^{n}\gamma_i(z)\,Q_i(t)$

gives $\beta_n^{(n)}/(z-a_{r-1})=\alpha_{n-1}(z)\,\beta_{n-1}^{(n-1)}$, $\gamma_n(z)\,\beta_{n-1}^{(n)}+\gamma_{n-1}(z)\,\beta_{n-1}^{(n-1)}=0$, $\beta_n^{(n)}/(z-a_r)=\gamma_n(z)\,\beta_n^{(n)}$. It follows that $\alpha_{n-1}(z)=\beta_{n-1}^{(n)}/\beta_{n-1}^{(n-1)}\cdot(z-a_{r-1})$, $\gamma_n(z)=1/(z-a_r)$, $\gamma_{n-1}(z)=-\beta_{n-1}^{(n)}/\beta_{n-1}^{(n-1)}\cdot(z-a_r)$. Thus we may write

$$\phi_n(t)=-\frac{1}{(z-a_r)}\,Q_n(t)-\frac{\beta_{n-1}^{(n)}}{\beta_{n-1}^{(n-1)}}\left[\frac{1}{(z-a_{r-1})}-\frac{1}{(z-a_r)}\right]Q_{n-1}(t)+Q(t,z), \tag{3.10}$$

where $Q(t,z)$ as a function of t belongs to $\mathscr{R}_{n-2}$. Consequently $\Phi_t(Q_{n-1}(t)\,\phi_n(t))=\beta_{n-1}^{(n)}\cdot(a_r-a_{r-1})/\beta_{n-1}^{(n-1)}\cdot(z-a_r)(z-a_{r-1})$. Application of Φ_t to the right side of Eq. (3.9) thus gives the right side of Eq. (3.8). ▮

THEOREM 3.5. *The following formula is valid for* $n=2,3,\ldots$,

$$\begin{aligned}&[(\zeta-a_r)(z-a_{r-1})\,Q_n(\zeta)\,Q_{n-1}(z)-(\zeta-a_{r-1})(z-a_r)\,Q_{n-1}(\zeta)\,Q_n(z)]\\&\quad=\frac{\beta_{n-1}^{(n)}}{\beta_{n-1}^{(n-1)}}(\zeta-z)(a_r-a_{r-1})\sum_{i=0}^{n-1}Q_i(z)\,Q_i(\zeta).\end{aligned} \tag{3.11}$$

Proof. Set $G(\zeta)=[(\zeta-a_r)(z-a_{r-1})\,Q_{n-1}(z)\,Q_n(\zeta)-(\zeta-a_{r-1})(z-a_r)\,Q_{n-1}(\zeta)\,Q_n(z)]$, and $F(\zeta)=G(\zeta)/(\zeta-z)$. We note that $F(\zeta)\in\mathscr{R}_{n-1}$, so that we may write $F(\zeta)=\sum_{i=0}^{n-1}\alpha_i(z)\,Q_i(\zeta)$, where $\alpha_i(z)=\langle F(\zeta),Q_i(\zeta)\rangle$. Let $i\leqslant n-1$. Since $(\zeta-a_r)\cdot(D_zQ_i)(\zeta)$ as a function of ζ belongs to $\mathscr{R}_{n-1}$, and $(\zeta-a_{r-1})\cdot(D_zQ_i)(\zeta)$ as a function of ζ belongs to $\mathscr{R}_{n-2}$, we conclude that $\Phi_\zeta((D_zQ_i)(\zeta)\cdot G(\zeta))=0$. Further we can write

$$\begin{aligned}\Phi_\zeta\left(\frac{G(\zeta)}{\zeta-z}\right)&=(z-a_{r-1})\,Q_{n-1}(z)\,\Phi_\zeta\left(\frac{(\zeta-a_r)\,Q_n(\zeta)-(z-a_r)\,Q_n(z)}{\zeta-z}\right)\\&\quad-(z-a_r)\,Q_n(z)\,\Phi_\zeta\left(\frac{(\zeta-a_{r-1})\,Q_{n-1}(\zeta)-(z-a_{r-1})(Q_{n-1}(z))}{\zeta-z}\right)\\&=(z-a_{r-1})(z-a_r)[Q_{n-1}(z)\,P_n(z)-Q_n(z)\,P_{n-1}(z)],\end{aligned}$$

by Lemma 3.1. Thus we have

$$\begin{aligned}\langle F,Q_i\rangle=\Phi_\zeta\left(\frac{Q_i(\zeta)}{\zeta-z}\cdot G(\zeta)\right)&=\phi_\zeta((D_zQ_i)(\zeta)\cdot G(\zeta))+Q_i(z)\,\Phi_\zeta\left(\frac{G(\zeta)}{\zeta-z}\right)\\&=(z-a_{r-1})(z-a_r)\,Q_i(z)[Q_{n-1}(z)\,P_n(z)-Q_n(z)\,P_{n-1}(z)].\end{aligned}$$

From Theorem 3.4 we now obtain $\alpha_i(z)=Q_i(z)(a_r-a_{r-1})\,\beta_{n-1}^{(n)}/\beta_{n-1}^{(n-1)}$, and the desired result follows from the equality $F(\zeta)=\sum_{i=0}^{n-1}\alpha_i(z)\,Q_i(\zeta)$. ▮

In the following, $T_n(z,a,b)$ shall denote the expression $aP_n(z)+bQ_n(z)$, where a and b are arbitrary complex numbers. Asterisk denotes complex conjugation.

THEOREM 3.6. *The following formula holds for* $n=2, 3, \dots,$

$$
\begin{aligned}
&[(\zeta-a_r)(z-a_{r-1})\,T_n(\zeta,\alpha,\beta)\,T_{n-1}(z,a,b)] \\
&\quad -(\zeta-a_{r-1})(z-a_r)\,T_{n-1}(\zeta,\alpha,\beta)\,T_n(z,a,b)] \\
&=\frac{\beta_{n-1}^{(n)}}{\beta_{n-1}^{(n-1)}}(a_r-a_{r-1})\left\{(\alpha b-a\beta)+(\zeta-z)\sum_{i=0}^{n-1}T_i(z,a,b)\,T_i(\zeta,\alpha,\beta)\right\}.
\end{aligned}
\tag{3.12}
$$

Proof. Subtraction of formula (3.11) from the same formula with z replaced by t gives

$$
\begin{aligned}
&(\zeta-a_r)\,Q_n(\zeta)[(t-a_{r-1})\,Q_{n-1}(t)-(z-a_{r-1})\,Q_{n-1}(z)] \\
&\quad -(\zeta-a_{r-1})\,Q_{n-1}(\zeta)[(t-a_r)\,Q_n(t)-(z-a_r)\,Q_n(z)] \\
&=\frac{\beta_{n-1}^{(n)}}{\beta_{n-1}^{(n-1)}}(a_r-a_{r-1})\sum_{i=0}^{n-1}Q_i(\zeta)[(\zeta-t)\,Q_i(t)-(\zeta-z)\,Q_i(z)].
\end{aligned}
$$

Division of this equality by $t-z$ and application of the functional Φ_t, Lemma 3.1 being used, gives the formula

$$
\begin{aligned}
&(\zeta-a_r)(z-a_{r-1})\,Q_n(\zeta)\,P_{n-1}(z)-(\zeta-a_{r-1})(z-a_r)\,Q_{n-1}(\zeta)\,P_n(z) \\
&=\frac{\beta_{n-1}^{(n)}}{\beta_{n-1}^{(n-1)}}\left\{(\zeta-z)(a_r-a_{r-1})\sum_{i=0}^{n-1}P_i(z)\,Q_i(\zeta)-(a_r-a_{r-1})\right\}.
\end{aligned}
\tag{3.13}
$$

In the same way as formula (3.13) is obtained from formula (3.11), we obtain from formula (3.13) the following formula

$$
\begin{aligned}
&(\zeta-a_r)(z-a_{r-1})\,P_n(\zeta)\,P_{n-1}(z)-(\zeta-a_{r-1})(z-a_r)\,P_{n-1}(\zeta)\,P_n(z) \\
&=\frac{\beta_{n-1}^{(n)}}{\beta_{n-1}^{(n-1)}}(\zeta-z)(a_r-a_{r-1})\sum_{i=1}^{n-1}P_i(z)\,P_i(\zeta).
\end{aligned}
\tag{3.14}
$$

By multiplying out the terms on the left side of formula (3.12) and using formulas (3.11), (3.13) (and its dual obtained by changing ζ and z) and (3.14) taking into account that $P_0=0$, we get formula (3.12). ∎

COROLLARY 3.7. *The following formula holds for* $n=2, 3, \dots,$ $\operatorname{Im} z \neq 0$, $w \in \mathbb{C}$,

$$
\begin{aligned}
&(z^*-a_r)(z-a_{r-1})\,T_n(z,1,w)^*\,T_{n-1}(z,1,w) \\
&\quad -(z^*-a_{r-1})(z-a_r)\,T_{n-1}(z,1,w)^*\,T_n(z,1,w) \\
&=\frac{\beta_{n-1}^{(n)}}{\beta_{n-1}^{(n-1)}}(a_r-a_{r-1})(z-z^*)\left\{-\frac{w-w^*}{z-z^*}+\sum_{i=0}^{n-1}|T_n(z,1,w)|^2\right\}.
\end{aligned}
\tag{3.15}
$$

Proof. Note that P_n and Q_n are R-functions with real coefficients. Therefore $Q_n(z)^* = Q_n(z^*)$, $P_n(z)^* = P_n(z^*)$. By setting $\zeta = z^*$, $a = \alpha = 1$, $b = w$, $\beta = w^*$ in formula (3.12), we obtain formula (3.15). ▮

4. Nested Discs and Stieltjes Transforms

For every complex number z outside the real axis and every complex number τ we define

$$f_n(z, \tau) = -\frac{P_n(z, \tau)}{Q_n(z, \tau)}. \tag{4.1}$$

We can then also write (cf. Lemma 3.2):

$$f_n(z, \tau) = -\frac{P_n(z) - \tau\,((z - a_{r-1})/(z - a_r))\,P_{n-1}(z)}{Q_n(z) - \tau\,((z - a_{r-1})/(z - a_r)\,Q_{n-1}(z)} \tag{4.2}$$

and

$$f_n(z, \tau) = -\frac{(z - a_r)\,P_n(z) - \tau(z - a_{r-1})\,P_{n-1}(z)}{(z - a_r)\,Q_n(z) - \tau(z - a_{r-1})\,Q_{n-1}(z)}. \tag{4.3}$$

We note that if $w = f(z, \tau)$, then

$$\tau = \frac{(z - a_r)[P_n(z) + w\,Q_n(z)]}{(z - a_{r-1})[P_{n-1}(z) + w\,Q_{n-1}(z)]}. \tag{4.4}$$

If $Q_n(z)$ is nonregular, then $f_n(z, \tau) = -\tau P_n(z)[1-k]/\tau Q_n(z)[1-k] = -P_n(z)/Q_n(z)$. In this case the linear fractional transformation $\tau \to f_n(z, \tau)$ is singular. If $Q_n(z)$ is regular, then $\beta^{(n)}_{n-1} \neq 0$. It follows from Theorem 3.4 that $P_n(z)\,Q_{n-1}(z) - P_{n-1}(z)\,Q_n(z) \neq 0$. The value of the determinant of the mapping $\tau \to f_n(z, \tau)$ is $((z - a_{r-1})/(z - a_r))[P_n(z)\,Q_{n-1}(z) - P_{n-1}(z)\,Q_n(z)]$, hence different from zero. Consequently the mapping $\tau \to f_n(z, \tau)$ is nonsingular in this case. The real axis is then mapped onto a circle $\Gamma_n(z)$. Let $\Delta_n(z)$ be the closed disc bounded by $\Gamma_n(z)$.

Theorem 4.1. *For every regular $Q_n(z)$ the following results hold true:*

(A) *The disc $\Delta_n(z)$ is given by*

$$w \in \Delta_n(z) \qquad \textit{iff} \quad \sum_{i=0}^{n-1} |T_i(z, 1, w)|^2 \leqslant \frac{w - w^*}{z - z^*}. \tag{4.5}$$

(B) *The radius $\rho_n(z)$ of the disc $\Delta_n(z)$ is given by*

$$\rho_n(z) = \left\{ |z - z^*| \cdot \sum_{i=0}^{n-1} |Q_i(z)|^2 \right\}^{-1}. \tag{4.6}$$

(C) *If $Q_m(z)$ is regular and $m > n$, then $\Delta_m(z) \subset \Delta_n(z)$.*

Proof. The point w belongs to $\Gamma_n(z)$ iff $w = f_n(z, \tau)$ for some $\tau \in \mathbb{R}$, i.e., iff $\operatorname{Re}(z - a_r)[P_n(z) + wQ_n(z)]/(z - a_{r-1})[P_{n-1}(z) + wQ_{n-1}(z)] = 0$, by (4.4). This condition is equivalent with

$$\operatorname{Re}(z - a_r)(z^* - a_{r-1}) \cdot [P_n(z) + wQ_n(z)] \cdot [P^*_{n-1}(z) + w^* Q^*_{n-1}(z)] = 0,$$

or equivalently

$$\begin{aligned}&(z - a_{r-1})(z^* - a_r)\, T_{n-1}(z, 1, w)\, T_n(z, 1, w)^* \\ &\qquad - (z - a_r)(z^* - a_{r-1})\, T_{n-1}(z, 1, w)^*\, T_n(z, 1, w) = 0.\end{aligned}$$

Thus $z \in \Gamma_n(z)$ iff $\sum_{i=0}^{n-1} |T_i(z, 1, w)|^2 = (w - w^*)/(z - z^*)$, by Corollary 3.7. A standard mapping argument shows that $\sum_{i=0}^{n-1} |T_i(z, 1, w)|^2 < (w - w^*)/(z - z^*)$ when w is inside $\Gamma_n(z)$. This proves (A).

A standard result on fractional linear transformations shows that

$$\rho_n(z) = \left| \frac{(z - a_r)(z - a_{r-1}) P_n(z) Q_{n-1}(z) - (z - a_r)(z - a_{r-1}) P_{n-1}(z) Q_n(z)}{(z - a_r)(z^* - a_{r-1}) Q_n(z) Q_{n-1})(z^*) - (z^* - a_r)(z - a_{r-1}) Q_{n-1}(z) Q_n(z^*)} \right|. \tag{4.7}$$

Substitution from Theorems 3.3 and 3.4 (with $\zeta = z^*$) leads to formula (4.6). This proves (B).

The result of (C) follows immediately from (A). ∎

We now define the set $\Delta_\infty(z)$ by

$$\Delta_\infty(z) = \cap \{\Delta_n(z) \colon Q_n \text{ is regular}\}. \tag{4.8}$$

It follows from Theorem 4.1 (C) that $\Delta_\infty(z)$ is a single point or a closed disc, with radius $\rho(z) = \{ |z - z^*| \sum_{i=0}^{\infty} |Q_i(z)|^2 \}^{-1}$. Since $\Delta_\infty(z)$ is a single point iff $\sum_{i=0}^{\infty} |Q_i(z)|^2 = \infty$ and since always $\sum_{i=0}^{\infty} |T_i(z, 1, w)|^2 < \infty$ when $w \in \Delta_\infty(z)$, we also have that $\Delta_\infty(z)$ is a single point iff $\sum_{i=0}^{\infty} |P_i(z)|^2 < \infty$.

THEOREM 4.2. *If $\Delta_\infty(z)$ is a disc for some z (outside the real axis), then $\Delta_\infty(\zeta)$ is a disc for every ζ (outside the real axis).*

Proof. If $\sum_{i=0}^{\infty} |Q_i(z)|^2 < \infty$, then by the aid of Lemma 3.3 it can be shown that also $\sum_{i=0}^{\infty} |Q_i(\zeta)|^2 < \infty$. The argument is almost exactly analogous to the proof of [1, Theorem 1.3.2]. ∎

In view of Theorem 4.2 we may use the terms *limit circle case* and *limit point case* without referencce to any particular point z.

A bounded, real-valued, nondecreasing function ψ is said to represent the functional Φ on a subspace $\mathscr{S}$ of $\mathscr{R}$ if $\Phi(f)=\int_{-\infty}^{\infty} f(t)\, d\psi(t)$ for every $f\in\mathscr{S}$. In particular the function ψ represents Φ on $\mathscr{R}$ iff it is a solution of the EHMP for the sequences $c_n^{(i)}$. (It is easily proved that when Φ is positive definite, then every bounded real-valued nondecreasing function representing Φ on an infinitely dimensional subspace of $\mathscr{R}$ has infinitely many points of increase, cf. [10, Proposition 5.5].) Similarly the function ψ represents Φ on $\mathscr{R}^0$ iff it is a solution of the relaxed EHMP.

We recall that the Stieltjes transform $\hat{\psi}$ of a bounded, real-valued, nondecreasing function ψ is given by

$$\hat{\psi}(z)=\int_{-\infty}^{\infty}\frac{d\psi(t)}{t-z}, \qquad \operatorname{Im} z\neq 0. \tag{4.9}$$

We define the subsets $\sum_z(s_1,..., s_p)$, $\sum_z^0(s_1,..., s_p)$, $\sum_z$, $\sum_z^0$ of the complex plane by

$$\textstyle\sum_z(s_1,..., s_p)=\{\hat{\psi}(z)\colon \psi \quad \text{represents} \quad \Phi \quad \text{on} \quad \mathscr{R}(s_1,..., s_p), \tag{4.10}$$

$$\textstyle\sum_z^0(s_1,..., s_p)=\{\hat{\psi}(z)\colon \psi \quad \text{represents} \quad \Phi \quad \text{on} \quad \mathscr{R}^0(s_1,..., s_p), \tag{4.11}$$

$$\textstyle\sum_z=\{\hat{\psi}(z)\colon \quad \psi \quad \text{represents} \quad \Phi \quad \text{on} \quad \mathscr{R}\}, \tag{4.12}$$

$$\textstyle\sum_z^0=\{\hat{\psi}(z)\colon \quad \psi \quad \text{represents} \quad \Phi \quad \text{on} \quad \mathscr{R}^0\}. \tag{4.13}$$

These sets are obviously convex (since $\alpha\psi_1+(1-\alpha)\,\psi_2$ is nondecreasing for $0\leqslant\alpha\leqslant 1$), and by a standard application of Helly's selection theorems they are also seen to be compact.

We recal the Gaussian quadrature formulas associated with quasi-orthogonal R-functions described in Section 2. For every $n=1, 2,...,$ and $\tau\in\mathbb{R}$ we define the function $\psi_{n,\tau}$ by

$$\psi_{n,\tau}(t)=\sum\{\lambda_{n,i}(\tau)\colon t_j(\tau)\leqslant t\}. \tag{4.14}$$

In the following let $s_1,..., s_p$ be as in Theorem 2.6. It follows from Theorem 2.6 that $\psi_{n,\tau}$ represents Φ on $\mathscr{R}^0(s_1,..., s_p)$ when $Q_n(t,\tau)$ is degenerate, and on $\mathscr{R}(s_1,..., s_p)$ when $Q_n(t,\tau)$ is non-degenerate.

As in Section 3 we set $f_n(z,\tau)=-P_n(z,\tau)/Q_n(z,\tau)$, and we assume that $\operatorname{Im} z\neq 0$, $\tau\in\mathbb{R}$.

THEOREM 4.3. *The following equality holds*:

$$\hat{\psi}_{n,\tau}(z) = f_n(z, \tau). \tag{4.15}$$

Proof. The function $\phi_n(t) = (Q_n(t, \tau) - Q_n(z, \tau)/(t-z)$ belongs to $\mathscr{R}^0(s_1,..., s_p)$ (the numbers $s_1,..., s_p$ are as in Theorem 2.6). By using Theorem 2.6 we therefore get $P_n(z, \tau) = \Phi_t(\phi_n) = \sum_{i=1}^{n} \lambda_{n,i}(\tau)(-Q_n(z, \tau))/(t_i - z) = -Q_n(t, \tau) \int_{-\infty}^{\infty} d\psi_{n,\tau}(t)/(t-z)$, and hence $f_n(z, \tau) = \int_{\infty}^{\infty} d\psi_{n,\tau}(t)/(t-z)$. ∎

THEOREM 4.4. *The following inclusions hold when $Q_n(z)$ is regular*:

$$\Delta_n(z) \subset \textstyle\sum_z^0(s_1,..., s_p) \qquad \textit{always}, \tag{4.16}$$

$$\Delta_n(z) \subset \textstyle\sum_z(s_1,..., s_p) \qquad \textit{when } Q_n(z) \textit{ is not bi-degenerate}. \tag{4.17}$$

Proof. Let $Q_n(z)$ be regular, so that $\Delta_n(z)$ and $\Gamma_n(z)$ are defined. Since $\psi_{n,\tau}(t)$ always represents Φ on $\mathscr{R}^0(s_1,..., s_p)$, it follows from Theorem 4.3 that every point on $\Gamma_n(z)$ except possibly one (corresponding to $\tau = \infty$) belongs to $\sum_z^0(s_1,..., s_p)$. Since $\sum_z^0(s_1,..., s_p)$ is convex and compact, we conclude that the inclusion $\Delta_n(z) \subset \sum_z^0(s_1,..., s_p)$ always holds. We saw in Section 2 that if $Q_n(z)$ is not bi-degenerate, then $Q_n(t, \tau)$ is degenerate for at most one value of τ. Since $\psi_{n,\tau}(t)$ represents Φ on $\mathscr{R}(s_1,..., s_p)$ whenever $Q_n(z, \tau)$ is non-degenerate, it follows from Theorem 4.3 that every point on $\Gamma_n(z)$ except possibly two belong to $\sum_z(s_1,..., s_p)$. As above we conclude that in this case the inclusion $\Delta_n(z) \subset \sum_z(s_1,..., s_p)$ holds. ∎

We have seen that when $Q_n(z)$ is nonregular, then the image $\{w: w = f_n(z, \tau), \ \tau \in \mathbb{R}\}$ reduces to a single point. The argument above shows that this point always belongs to $\sum_z^0(s_1,..., s_p)$, and to $\sum_z(s_1,..., s_p)$ when $Q_n(z)$ is non bidegenerate.

THEOREM 4.5. *The following inclusion holds when $Q_n(z)$ is regular, $\sigma_1 = \cdots = \sigma_r = 2q+2$, $\sigma_{r+1} = \cdots = \sigma_p = 2q$*:

$$\textstyle\sum_z(\sigma_1,..., \sigma_p) \subset \Delta_n(z). \tag{4.18}$$

Proof. Let $w \in \sum_z(\sigma_1,..., \sigma_p)$. There exists a bounded, nondecreasing function $\psi(t)$ which represents Φ on $\mathscr{R}(\sigma_1,..., \sigma_p)$ such that $\psi(z) = w$. Then the inner products defined by ψ and by Φ coincide on $\mathscr{R}_{n-1}$, hence the system $\{Q_0, ..., Q_{n-1}\}$ is orthonormal with respect to ψ. We set $f(t) = 1/(t-z)$. Then

$$\langle f, f \rangle = \int_{-\infty}^{\infty} \frac{d\psi(t)}{|t-z|^2} = \frac{1}{(z-z^*)} \left[\int_{-\infty}^{\infty} \frac{d\psi(t)}{t-z} - \int_{-\infty}^{\infty} \frac{d\psi(t)}{t-z^*} \right] = \frac{w-w^*}{z-z^*}.$$

Also $\langle f, Q_i \rangle = \int_{-\infty}^{\infty} (Q_i(t) - Q_i(z))/(t-z) \; d\psi(t) + Q_i(z) \int_{-\infty}^{\infty} d\psi(t)/(t-z) = P_i(z) + wQ_i(z) = T_i(z, 1, w)$. Bessel's inequality then reads $\sum_{i=0}^{n-1} |T_i(z, 1, w)|^2 \leqslant (w-w^*)/(z-z^*)$, and the desired conclusion follows from Theorem 4.1(A). ∎

THEOREM 4.6. *The following inclusions hold*:

$$\sum_z \subset \Delta_\infty(z) \subset \sum_z^0. \tag{4.19}$$

Proof. The inclusion $\sum_z \subset \Delta_\infty(z)$ follows immediately from Theorem 4.6, since $\sum_z \subset \sum_z(\sigma_1, ..., \sigma_p)$ for all n.

Let $w \in \partial\Delta_\infty(z)$, and let $\{n(k)\}$ be a sequence of indices for which $Q_{n(k)}(z)$ are regular (cf. Theorem 2.5). There exists by Theorem 4.3 a sequence $\{\tau(n(k))\}$, $\tau(n(k)) \in \mathbb{R}$, such that $\psi_{n(k),\tau(n(k))}$ represents Φ on $\mathscr{R}^0(s_1, ..., s_p)$ and $\{\hat{\psi}_{n(k),\tau(n(k))}(z)\}$ converges to w. Now $\int_{-\infty}^{\infty} d\psi_{n(k),\tau(n(k))}(t) = \sum_{i=1}^{\nu} \lambda_{n,i}(\tau(n(k))) \leqslant \lambda_{n,0}(\tau(n(k))) + \sum_{i=1}^{\nu} \lambda_{n,i}(\tau(n(k))) = \Phi(1)$, so the sequence $\{\psi_{n(k),\tau(n(k))}\}$ is uniformly bounded. A standard argument involving Helly's selection theorems shows that there exists a bounded, nondecreasing function ψ which represents Φ on $\mathscr{R}^0$ such that $\hat{\psi}(z) = w$. Since ψ automatically has infinitely many points of increase (cf. the comments before the introduction of the Stieltjes transform, formula (4.9)), we conclude that $\partial\Delta_\infty(z) \subset \sum_z^0$. The inclusion $\Delta_\infty(z) \subset \sum_z^0$ now follows by convexity of $\sum_z^0$. ∎

THEOREM 4.7. *If there are infinitely many indices n for which $Q_n(z)$ is both regular and not bi-degenerate, then*

$$\sum_z = \Delta_\infty(z).) \tag{4.20}$$

Proof. Let $\{(k)\}$ be a sequence of indices for which $Q_{n(k)}(z)$ are regular and not bi-degenerate. Let $w \in \partial\Delta_\infty(z)$. Then $\tau((k))$ may be chosen such that the function $\psi_{n(k),\tau(n(k))}$ in the proof of Theorem 4.6 represents Φ on the whole of $\mathscr{R}(s_1, ..., s_p)$, and the limit function ψ represents Φ on $\mathscr{R}$. It follows that $w \in \sum_z$. Thus $\partial\Delta_\infty(z) \subset \sum_z$. ∎

THEOREM 4.8. *The following results hold*:

(A) *In the limit point case* EHMP *has a unique solution.*

(B) *In the limit circle case the relaxed* EHMP *has infinitely many solutions.*

(C) *If there are infinitely many indices n for which* $Q_n(z)$ *is both regular and not bi-degenerate, then* EHMP *has a unique solution exactly in the limit point case.*

Proof. Follows immediately from Theorem 4.6, Theorem 4.7, and the essential uniqueness of the inverse Stieltjes transform. ∎

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